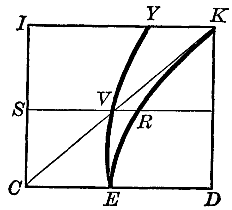
row's credit. For example Barrow proves the theorem (Lecture XII, Appendix I, 9):



"Let ERK be an equilateral hyperbola (that is, one having equal axes), and let the axes be CED, CI; also let KI, KD be ordinates to these; let EVY be a curve such that, when any point R is taken at random on the hyperbola, and a straight line RVS is drawn parallel to DC, then SR CE, SV are in continued proportion; join CK; then the space $CEYI$ will be double the hyperbolic sector KCE."

The adjoining figure is our own; Child gives none. The determination of the area $CEYI$ by calculus involves the formula

$$\int_0^y \frac{dy}{\sqrt{a^2+y^2}} = \log \frac{y+\sqrt{a^2+y^2}}{a}.$$

Hence Child claims that Barrow possessed this formula in geometrical garb. Accordingly, the geometrical process of integrating

$$\int_0^y \frac{dy}{\sqrt{a^2+y^2}}$$

would be to construct the hyperbola $x^2 - y^2 = a^2$ and from it the curve EVY, yielding $CEYI$ as the area representing the definite integral. By the same argument it may be claimed that when Dinostratus of old used the quadratrix in the quadrature of the circle, he worked out the part of the integral calculus contained in the formula $\int_0^a \sqrt{a^2 - x^2}dx = \frac{1}{4}\pi a^2$. Dinostratus and Barrow were clever men, but it seems to us that they did not create what by common agreement of mathematicians has been designated by the term differential and integral calculus. Two processes yielding equivalent results are not necessarily the same. It appears to us that what can be said of Barrow is that he worked out a set of geometric theorems suggesting to us constructions by which we can find lines, areas and volumes whose magnitudes are ordinarily found by the analytical processes of the calculus. But to say that Barrow invented a differential and integral calculus is to do violence to the habit of mathematical thought and expression of over two centuries. The invention rightly belongs to Newton and Leibniz.

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QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

The discussion given below is closely related to Professor Moritz's article "On the Construction of Certain Curves Given in Polar Coördinates" published

in the May, 1917, issue of this MONTHLY, pp. 213–220. Professor Moritz discussed "cyclic-harmonic" curves resulting from the motion of a point which has simple harmonic motion along a line, while at the same time this line rotates with a constant angular velocity about one of its fixed points. Professor Rigge gives below a very complete discussion of a method for tracing a cardioid as the resultant of a simple harmonic motion along a line and a uniform angular motion about a point not in that line.

CONCERNING A NEW METHOD OF TRACING CARDIOIDS.

By WILLIAM F. RIGGE, S.J., Creighton University, Omaha, Nebr.

A cardioid may under certain conditions be traced by a point when its motion is the resultant of a rectilinear simple harmonic movement and a uniform angular one of the same period. The harmonic motion may be obtained from any plane mechanical contrivance such as a revolving crank and a sliding slotted bar, while the angular one is best furnished by a disk rotating under the pen.

Fig. 1 will illustrate the definition just given as well as the conditions to be mentioned presently. The point A is the center of the disk which rotates in a clockwise direction with uniform angular speed. If the disk did not rotate, the tracing pen would move over the line EG parallel to the Y axis with simple harmonic motion, so that its distance from R, the middle point, would at any moment be the sine of the phase, the amplitude RE or RG being taken as the unit of our scale in this investigation. But when the disk does rotate, the combination of the rectilinear motion of the pen with the rotary one of the disk causes the pen B to trace the cardioid $BCKQZFB$ (R is only by accident on the curve), provided the following conditions are observed.

Conditions to be Observed.—First, the pen may be set down on the disk as at B at any initial phase α of its rectilinear harmonic motion. In Fig. 1 this initial phase α is taken as 52°, so that $RB = \sin 52°$, RE, as said, being unity.

Second, the point B, at which the pen is set down on the disk, must be on the unit circle whose center O is on the Y axis and whose distance from A, the center of the disk, is the sine of the phase $OA = \sin\alpha = \sin 52° = BR$. The angle AOB we will call β, the *starting* angle, and the circle just mentioned the *starting* circle. In Fig. 1 β is equal to 77°.

Study of the Conditions.—It is to be noted first that α and β are independent variables and may have any values whatever.

Secondly, the initial phase α fixes the center of the starting circle O on the axis of Y, so that O is in the same direction from A that B is from R, and $OA = BR$. It also fixes the center D of what we will call the *cusp*-circle, because it is the locus of the cusp of the cardioid that can be generated with the given initial phase α. The radius of this small circle is one half that of the starting circle, or one half of our chosen unit, its circumference passes through A and O, and it is internally tangent to the large circle at S on the axis of X, so that the angle $ASO = \alpha$. This puts S to the left of A when $\cos\alpha$ is positive, and to the right